Dielectric response of particle beams to periodic focusing

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The dielectric response of a charged particle beam to a periodic focusing field enhances the effective focusing strength, reducing the matched beam radius and affecting the motion of halo particles. The change in the effective focusing strength is found for a uniform-density beam with a diffuse halo in a quadrupole channel, giving increases of 2% to 8% for some typical examples. These changes are important for both the production and behavior of halos in intense, high-energy beams, in which fractional current losses as small as 10^{-8}/m can result in radioactivation. [S1063-651X(96)04211-0]

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The effective focusing strength of a periodic channel is an important factor for accelerator applications requiring high beam intensities, such as heavy ion inertial fusion, radioactive waste transmutation, spallation neutron sources, tritium production, and muon production. Limiting currents have been found in the past using the smooth approximation [1] to find the effective focusing strength of a periodic channel which, along with the aperture, determines the current that can be transported through a given channel [2]. Accurate knowledge of the effective focusing strength is also important for matching. Transverse mismatch has been shown to be an important cause of halo production and the resulting particle losses [3–5]. Fractional current losses as small as 10^{-8}/m can result in radioactivation, inhibiting routine maintenance [6]; this can also be the limiting factor in the transport of intense, high-energy beams [7].

The dielectric response of a plasma to the periodic field of a Paul trap was recently shown to enhance the effective focusing strength of the trap [8]. The dielectric response of a beam to a periodic focusing field is shown here to increase the effective focusing strength of the channel, by an amount that depends on the shape of the beam, the type of focusing, and the ratio of the plasma frequency of the beam, ω_p, to the frequency of the focusing, ω. The dielectric response and the fractional change in the effective focusing strength are found for a uniform-density continuous beam with a diffuse halo and for a uniform-density ellipsoidal (bunched) beam, both in a quadrupole channel. The increase in the effective focusing strength results in a higher transverse phase advance per period, a higher average beam density, and a lower average beam radius. Since accurate matching is important for beam applications requiring low losses, the effect of the dielectric response on the matched beam parameters can be important for the applications listed above.

A beam in a periodic focusing channel experiences a fluctuating electric field $\vec{E}_f(\vec{r},s_0)$, which consists of the fluctuating component of the focusing field, $\vec{E}_{cf}(\vec{r},s_0)$, and small fluctuations in the space charge field, $\vec{E}_{sf}(\vec{r},s_0)$. The position relative to the center of the beam is $\vec{r}$, and the focusing is periodic in s, the longitudinal distance along the channel. Although particles with different longitudinal positions within the beam are at different phases in the periodic field, it is assumed that the effects of this are negligible so that the fluctuating fields can be written as periodic functions of the longitudinal position of the beam center along the channel, s_0. The focusing field and the space charge field are each divided into two parts so that the fluctuating components have an average value of zero and the steady-state components vary slowly or not at all with s_0.

The frequency of the focusing is $\omega = 2\pi\nu_B/S$, where S is the period of the focusing along the longitudinal direction and ν_B is the beam velocity. In general there are three periods (S_x, S_y, and S_z) and three frequencies (ω_x, ω_y, and ω_z), one for each of three directions in Cartesian coordinates (x and y are transverse and z is parallel to the beam axis; for most practical applications $S_x = S_y$). The focusing field can be the result of electrostatic or magnetic quadrupole lenses, induction-acceleration gaps, and magnetic solenoids (if the beam is considered in the Larmor frame). It can also be the result of focusing by electromagnetic fields which are periodic in time and space, as in the case of radio-frequency quadrupole (RFQ) focusing. The focusing field is written as an electric field with the approximation that particle motion in the beam frame is nonrelativistic, so that magnetic focusing can be represented by equivalent electrostatic fields. The force resulting from the magnetic field of the beam is included in the self-electric-field (the space charge field) with the same approximation. Unless otherwise stated, all quantities are considered in the laboratory frame. With RFQ focusing and induction-acceleration gaps, it is assumed that acceleration along the longitudinal direction is slow enough that it can be treated as adiabatic, and that the beam is in phase with the time-varying field so that the focusing field can be treated as periodic only in longitudinal distance along the channel.

The effective focusing field can be found from the average field of a particle due to its motion in the periodic field [9]. The motion of a particle in the periodic field is first found with the fluctuating field as a function of position fixed at $\vec{E}_f(\vec{r},s_0) = \vec{E}_f(\vec{r}_0,s_0)$, where $\vec{r}_0$ is the position of the particle averaged over a period. The resulting particle position is $\vec{r}_0 + \delta\vec{r}$; the first-order variation in the position of the particle resulting from the fluctuating field is $\delta\vec{r}$. The effective field that results from the fluctuating fields is then found to first order from

$$\vec{E}_{\mathrm{eff}} = \langle \vec{E}_f(\vec{r},s_0)\rangle \approx \langle (\delta\vec{r}\cdot\vec{\nabla}_0)\vec{E}_f(\vec{r}_0,s_0)\rangle, \tag{1}$$

where $\vec{\nabla}_0$ is the gradient with respect to $\vec{r}_0$ and the brackets represent averages over a focusing period, defined for an arbitrary function h by

$$\langle h(s_0)\rangle \equiv \frac{1}{S}\int_s^{s+S} h(s_0)\,ds_0. \tag{2}$$

The effective field of Eq. (1) has previously been derived for a Paul trap without space charge [9] and for a periodic focusing channel without space charge fluctuations [10]. Solving for $\delta\vec{r}$ from the fluctuating field and substituting into Eq. (1) gives an effective field of

$$\vec{E}_{\mathrm{eff}} \approx \frac{q}{\gamma m \nu_B^2}\left\langle\left(\int_0^{s_0}\int_0^{s_0'}\vec{E}_f(\vec{r}_0,s_0'')ds_0''ds_0'\cdot\vec{\nabla}_0\right)\vec{E}_f(\vec{r}_0,s_0)\right\rangle, \tag{3}$$

where q and m are, respectively, the particle charge and mass, and $\gamma=(1-\nu_B^2/c^2)^{-1/2}$ is the relativistic factor. In a quadrupole channel the steady-state component of the transverse focusing field is zero, so the field of Eq. (3) is the total effective focusing field. For transverse focusing by solenoids or longitudinal focusing by induction-acceleration gaps, the steady-state component of the focusing field is typically much larger than the effective field of Eq. (3), so that the dielectric response, which affects only the fluctuating component of the field, has much less effect than in a quadrupole channel with the same frequency and focusing strength.

The dielectric response occurs through the effect of space charge fluctuations on $\vec{E}_f(\vec{r}_0,s_0)$. This will be found first for the core and halo of a uniform-density continuous beam with a diffuse halo in a quadrupole channel with average axial symmetry. The dielectric response will then be considered for the core of a uniform-density ellipsoidal (bunched) beam in a quadrupole channel with average axial symmetry.

The electric field in a transverse direction (x) of a continuous, uniform elliptic beam with current I and velocity ν_B is [11]

$$E_{sx}=\frac{Ix}{\pi\varepsilon_0\gamma^2\nu_B x_m(x_m+y_m)}, \tag{4}$$

where x_m and y_m are, respectively, the beam envelopes in the x and y directions, and ε_0 is the permittivity of free space. In a quadrupole channel which has average axial symmetry, the fluctuations in the two transverse directions have the same magnitude and functional form, and are out of phase by π. The beam envelopes can then be written as $x_m=x_{m0}+\delta x_m$ and $y_m=x_{m0}-\delta x_m$, where x_{m0} is nearly independent of s_0 and δx_m has an average value of zero. The field of Eq. (4) can then be split into a steady-state component and a fluctuating component with a linear expansion in δx_m. The resulting fluctuating field component is

$$E_{sfx}=\frac{-I\,\delta x_m x}{2\pi\varepsilon_0\gamma^2\nu x_{m0}^3}. \tag{5}$$

Using Eq. (5), setting $E_{fx}=E_{sfx}+E_{cfx}$, where E_{fx}, E_{sfx}, and E_{cfx} are, respectively, the x components of the fluctuating parts of the effective focusing field, the space charge field, and the focusing field, and solving for δx_m gives

$$\vec{E}_{fx}=\vec{E}_{cfx}/\varepsilon, \tag{6}$$

in which ε (by definition) is the dielectric constant. The dielectric constant for this case is

$$\varepsilon=1-\Gamma\,\frac{\omega_p^2}{\omega^2}, \tag{7}$$

in which $\Gamma=\frac{1}{2}$, $\omega_p=(q^2n_s/\varepsilon_0\gamma m)^{1/2}$ is the plasma frequency, and n_s is the particle number density. Equation (7) will be used for other types of beams and for halos with different values for Γ, depending on the geometry.

In deriving Eqs. (6) and (7) it is assumed that $\omega_p^2/\omega^2\ll 1$, and that fluctuations in the focusing fields and space charge fields occur sinusoidally with the same frequency. For most focusing channels the fluctuating component of the focusing field is not a sinusoidal function of longitudinal distance along the channel. In order to define the dielectric constant, the fluctuations are approximated as sinusoidal functions of s_0. Small deviations in the functional form are assumed not to have a significant effect on the dielectric response of the beam.

Since $\varepsilon<1$, Eq. (6) represents an enhancement of the periodic focusing field for a continuous beam. This effect results from the fact that the beam has maxima in its extent along any axis, and minima in the magnitude of its space charge field, at longitudinal positions along the channel where the focusing field along that axis is at a maximum. Likewise, the beam has maxima in the magnitude of its space charge field where the focusing field is at a minimum. Fluctuations in the space charge field are therefore correlated with the focusing so that they enhance the effective focusing field.

Substituting Eq. (6) into Eq. (3) leads to the conclusion that the effect of the dielectric response of the beam is to increase the effective transverse focusing field of a quadrupole channel by the factor $1/\varepsilon^2$. For example, a continuous beam in a quadrupole channel with $\omega_p/\omega=0.2$ has a dielectric constant of 0.98. The dielectric response increases the effective focusing field of this channel by about 4%.

The same technique can be used to find the effect of the dielectric response on halo particles surrounding the uniform-density core of a continuous beam. The model of a uniform-density continuous beam core that is mismatched in a continuous (nonperiodic) focusing channel has been used to study the evolution of halo particles [5], in which variations in the space charge field resulting from the oscillating core were found to drive some particles to larger radii. Here, the effect on the effective focusing strength is found from oscillations of the space charge fields for a matched beam in a periodic channel. The same result applies to a mismatched beam in a periodic channel if the frequency of the mismatch oscillations is much less than ω.

The beam has average axial symmetry, and variations in x_m and y_m are out of phase by π. The dielectric response of halo particles arises from the periodic motion of the particles relative to the beam axis, and also from the periodic variations in the shape of the core. The electric field along the x direction outside of a continuous, uniform-density elliptic beam core is [11]

$$E_{sx}=\frac{I}{4\pi\varepsilon_0\gamma^2\nu_B}\int_t^\infty \frac{2x}{(x_m^2+t')^{3/2}(y_m^2+t')^{1/2}}\,dt', \quad (8)$$

where t is defined by $x^2/(x_m^2+t)+y^2/(y_m^2+t)=1$. The position of a halo particle is written as $(x,y)=(x_0+\delta x, y_0+\delta y)$, and the envelopes are again $x_m=x_{m0}+\delta x_m$ and $y_m=x_{m0}-\delta x_m$. The self-electric-field can then be written in terms of a steady-state component and a fluctuating component with linear expansions in the fluctuating quantities. Solving for the resulting particle motion by the same method as in the previous case, the fluctuating field is again described by Eqs. (6) and (7). In this case

$$\Gamma=\frac{x_{m0}^2(x_0^2-y_0^2)}{(x_0^2+y_0^2)}+\frac{x_{m0}^4(3y_0^2-x_0^2)}{(x_0^2+y_0^2)^3}. \quad (9)$$

For example, with $x_0=1.5x_{m0}$ and $y_0=0$, $\omega_p/\omega=0.2$ gives a dielectric constant of approximately 0.99. The dielectric response increases the effective focusing field at this location by about 2%.

The same method will now be used for the core of a bunched beam with average axial symmetry, which is taken as a uniform-density ellipsoid. The electric field in a transverse direction (x) inside a uniform ellipsoid without images is [11]

$$E_{sx}=\frac{3Qx}{8\pi\varepsilon_0\gamma}\int_0^\infty \frac{dt}{(x_m^2+t)^{3/2}(y_m^2+t)^{1/2}(\gamma^2z_m^2+t)^{1/2}}, \quad (10)$$

where z_m is the beam envelope in the z direction and Q is the total charge of each bunch. The envelope fluctuation in the longitudinal direction is typically either out of phase with the transverse fluctuations by $\pi/2$ or it has a different (and nonresonant) frequency from the transverse fluctuations; either way it can be ignored in finding the effective transverse focusing. The electric field of Eq. (10) can then be split into a steady-state component and a fluctuating component with a linear expansion in δx_m. The remaining integral is solvable analytically, resulting in a fluctuating field component of

$$E_{sfx}=\frac{-3Q\,\delta x_m x}{4\pi\varepsilon_0 x_{m0}^3\gamma^2 z_{m0}}\left[\frac{1}{2\xi^2}-\frac{3(1-\xi^2)}{4\xi^4}+\frac{3(1-\xi^2)^2\ln(\{1+\xi\}/\{1-\xi\})}{8\xi^5}\right], \quad (11)$$

where $\xi=(1-x_{m0}^2/\gamma^2z_{m0}^2)^{1/2}$ is the eccentricity of the bunch in the beam frame. For a bunch that is spherical in the beam frame ($\gamma z_{m0}=x_{m0}$), Eq. (11) becomes $E_{sfx}=-3Q\,\delta x_m x/(10\pi\varepsilon_0 x_{m0}^4)$. The same method as in the previous cases results again in Eqs. (6) and (7), in which Γ equals the quantity in square brackets in Eq. (11). For the special case in which the bunch is spherical in the beam frame, $\Gamma=0.4$. For example, a beam with an aspect ratio of $\gamma z_{m0}/x_{m0}=2$ (for which Γ is approximately 0.4) in a quadrupole channel with $\omega_p/\omega=0.3$, has a dielectric constant of approximately 0.964. The dielectric response increases the effective focusing field of this channel by about 8%.

The envelope equations [12] can be used to relate ω_p/ω to the transverse space charge tune depression (k_x/k_{x0}) and the phase advance per period (σ_{x0}), giving

$$\frac{\omega_p}{\omega}=\left(1-\frac{k_x^2}{k_{x0}^2}\right)^{1/2}\frac{\sigma_{x0}}{\pi\sqrt{2g_r}}, \quad (12)$$

in which $g_r=1-gx_m^2/2\gamma^2z_m^2$ is the radial geometry factor [13]. g is the geometry factor, which is a function only of the aspect ratio of the bunch, $\gamma z_m/x_m$, when image fields are negligible; it is a function also of the pipe radius when image fields are significant [13]. Without image fields, g can be approximated as $2\gamma z_m/3x_m$ when $1\leqslant\gamma z_m/x_m\leqslant 4$ with about 10% accuracy. Equation (12) applies for a continuous beam with $g_r=1$. The first example of a continuous beam with $\omega_p/\omega=0.2$ could therefore correspond to $k_x/k_{x0}=0.5$ and $\sigma_{x0}=59°$. The example of a bunched beam with $\gamma z_m/x_m=2$ and $\omega_p/\omega=0.3$ could correspond to $k_x/k_{x0}=0.5$ and $\sigma_{x0}=80°$.

Comparison will now be made between two uniform-density beams with the same energy, current, space charge tune depression, and aspect ratio, one in a periodic quadrupole channel and one in a channel with continuous focusing. Both channels have the same effective focusing strength in the absence of space charge. With space charge, the effective focusing strength of the periodic channel is increased over that of the continuous channel, resulting in a larger phase advance per period, a higher average beam density, and a smaller average beam radius.

The envelope equations for a matched beam in a continuous channel [12] can be used to derive $\omega_p^2=2\nu_B k_{x0}^2(1-k_x^2/k_{x0}^2)^{1/2}$. With a fixed space charge tune depression, the square of the plasma frequency of a uniform-density beam is therefore proportional to the focusing strength of the channel, k_{x0}^2. Using Eqs. (3) and (6), the dielectric response in the quadrupole channel results in an effective focusing strength of k_{x0}^2/ε^2. Comparing the plasma frequency of the beam in the continuous focusing channel (ω_{pc}) to that of the beam in the quadrupole channel (ω_p) then leads to

$$\Gamma\left(\frac{\omega_p}{\omega}\right)^3-\frac{\omega_p}{\omega}+\frac{\omega_{pc}}{\omega}=0. \quad (13)$$

An approximate analytic solution to Eq. (13) is

$$\frac{\omega_p}{\omega}\approx\frac{\omega_{pc}}{\omega}\left(\frac{1-(3-\Gamma)\omega_{pc}^2/\omega^2}{1-3\omega_{pc}^2/\omega^2}\right), \quad (14)$$

which is always accurate to within 1% for $0<\omega_{pc}/\omega<0.3$ and $-0.3<\Gamma<1$; this was found by a method similar to the one used previously for the approximate solution to a fourth-order polynomial [14].

Using the example of a bunched beam in a quadrupole channel with $\gamma z_m/r_m=2$, $k_x/k_{x0}=0.5$, and $\omega_p/\omega=0.3$, from Eq. (13) the plasma frequency of the beam in the continuous channel (ω_{pc}) corresponds to $\omega_{pc}/\omega=0.29$. From Eq. (12), $\sigma_{x0}=80°$ in the quadrupole channel and 77° in the continuous channel. The corresponding matched beam radius is approximately 4% lower in the quadrupole channel.

Reducing the frequency of the focusing increases the dielectric response and increases the effective focusing strength for a uniform-density beam, but also results in greater oscillations of the matched beam envelope. For applications in which current loss into the conducting channel is an important factor, the increase in the magnitude of the envelope oscillations as the focusing frequency is decreased could lead to greater particle losses even as the effective focusing field on the beam core is enhanced.

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